SUPER-MAGIC COMPLETE k-PARTITE HYPERGRAPHS

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ABSTRACT.

We deal with complete k-partite hypergraphs and we show that for all $k \geq 2$ and $n \neq 2, 6$ its hyperedges can be labeled by consecutive integers $1, 2, \ldots, n^k$ such that the sum of labels of the hyperedges incident to $(k-1)$ particular vertices is the same for all $(k-1)$-tuples of vertices from $(k-1)$ independent sets.

By a *complete k-partite hypergraph* $\mathbf{H}_n^k$ we mean a hypergraph with kn vertices divided into k independent sets each with n vertices and n^k hyperedges having exactly k vertices. (Note. We obtain $\mathbf{H}_n^k$ from a complete k-partite graph $\mathbf{K}_{n,n,\ldots,n}$ by replacing all the edges of its every complete subgraph $\mathbf{K}_k$ by a hyperedge.) A hypergraph $\mathbf{H}_n^k$ is *magic* if the hyperedges can be labeled with different positive integers such that the sum of labels of the hyperedges incident to $(k-1)$ particular vertices is the same for all $(k-1)$-tuples of vertices from $(k-1)$ independent sets. Moreover, if the labels are consecutive integers $1, 2, \ldots, n^k$ then $\mathbf{H}_n^k$ is called *super-magic*. A super-magic hypergraph $\mathbf{H}_n^k$ we denote by $\mathbf{M}_n^k$ and its hyperedge $(i_1, i_2, \ldots, i_k)$ with its label we denote by $\mathbf{m}(i_1, i_2, \ldots, i_k)$. In a similar way we can define a *magic* (or *super-magic*) hypergraph and its special case a *magic* (or *super-magic*) graph.

Magic graphs were introduced by J.Sedláček. Necessary and sufficient conditions for the existence of a magic graph can be found in [4] and [5]. B.M.Stewart [7] has proved that for all $n \neq 0 \mod 4$ and $n > 5$ the complete graph $\mathbf{K}_n$ is super-magic. It is easy to see that the classic concept of a magic square corresponds to the fact that the complete bipartite graph $\mathbf{K}_{n,n}$ is super-magic for all $n \neq 2$. J.Sedláček [6] considered the graph $\mathbf{M}_{2n}$ (also called the Möbius ladder) and constructed a super-magic labeling for odd $n > 3$. Super-magic labelings for some classes of regular graphs of degree 4 were described in [1] and [3].

Super-magic complete bipartite graphs $\mathbf{K}_{n,n}$ generalize to super-magic complete k-partite hypergraphs $\mathbf{M}_n^k$. The aim of this paper is to prove the following theorem.

Theorem.
If $n \neq 2, 6$ and $k \geq 2$ are positive integers, then the complete k-partite hypergraph $\mathbf{H}_n^k$ is super-magic.

Before we prove our result we consider Latin squares and a Latin hypergraph.

A *Latin square* $\mathbf{R}_n|\mathbf{r}(i,j);\ 1 \le i,j \le n|$ *of order* n is a square matrix of order n such that every row and column is a permutation of the set of natural numbers $\{1,2,\dots,n\}$. Two Latin squares $\mathbf{R}_n|\mathbf{r}(i,j)|$ and $\mathbf{S}_n|\mathbf{s}(i,j)|$ of order n are called *orthogonal*, if all n^2 ordered pairs $[\mathbf{r}(i,j),\mathbf{s}(i,j)]$, where $i,j \in \{1,2,\dots,n\}$, are different. In [2] it is proved that two orthogonal Latin squares of order n exist if and only if $n \ne 2,6$. We will use this statement to prove our theorem. A hypergraph $\mathbf{H}_k^n$ is called *Latin* if the hyperedges are labeled by integers $1,2,\dots,n$ such that labels of the hyperedges incident to $(k-1)$ particular vertices are integers $1,2,\dots,n$. A Latin hypergraph $\mathbf{H}_n^k$ we denote by $\mathbf{U}_n^k$ and its hyperedge $(i_1,i_2,\dots,i_k)$ with its label we denote by $\mathbf{u}(i_1,i_2,\dots,i_k)$.

On Figure are the labels of hyperedges of $\mathbf{M}_3^4$. In the $[3(b-1)+d]$-th row and the $[3(a-1)+c]$-th column is the label $\mathbf{m}(a,b,c,d)$ of the hyperedge which joins the a-th vertex of the first part of vertices, the b-th vertex of the second part, the c-th vertex of the third part and the d-th vertex of the fourth part.

46	8	69	17	78	28	60	37	26
62	42	19	51	1	71	10	80	33
15	73	35	55	44	24	53	6	64
59	39	25	48	7	68	16	77	30
12	79	32	61	41	21	50	3	70
52	5	66	14	75	34	57	43	23
18	76	29	58	38	27	47	9	67
49	2	72	11	81	31	63	40	20
56	45	22	54	4	65	13	74	36

FIGURE

This labeling of $\mathbf{M}_3^4$ was made using the following formula (it is true for all odd n and $k=4$)

$$\begin{aligned}\mathbf{m}(i_1,i_2,i_3,i_4) = {} & [(i_1 - i_2 + i_3 - i_4 + \tfrac{n-1}{2}) \mod n]n^3 \\ & + [(i_1 - i_2 + i_3 + i_4 - \tfrac{n+3}{2}) \mod n]n^2 \\ & + [(i_1 - i_2 - i_3 - i_4 + \tfrac{3n+1}{2}) \mod n]n \\ & + [(i_1 + i_2 + i_3 + i_4 - \tfrac{3n+5}{2}) \mod n] + 1.\end{aligned}$$

This formula was derived from the construction in the proof of our theorem.

Proof of the theorem. A super-magic hypergraph $\mathbf{M}_1^k$ has only one edge. Just as a magic square of order 2 does not exist $\mathbf{M}_2^k$ does not exist either and therefore we suppose that $n \ge 3$

We prove the theorem by a construction of $\mathbf{M}_n^k$ for all integers $3 \le n \ne 6$ and $k \ge 3$. We use mathematical induction on k. $\mathbf{U}_n^2$ is a Latin square of order n and $\mathbf{M}_n^2$ is a super-magic complete bipartite graph $\mathbf{K}_{n,n}$.

We suppose that $\mathbf{U}_n^{k-1}|\mathbf{u}(i_1,i_2,\dots,i_{k-1})|$ and $\mathbf{M}_n^{k-1}|\mathbf{m}(i_1,i_2,\dots,i_{k-1})|$ are already constructed. We define a Latin hypergraph $\mathbf{U}_n^k|\mathbf{u}(i_1,i_2,\dots,i_k)|$ and a super-magic complete

k-partite hypergraph $\mathbf{M}_n^k|\mathbf{m}(i_1,i_2,\ldots,i_k)|$ for all $1\le i_1,i_2,\ldots,i_k\le n$ by the following relations

$$\mathbf{u}(i_1,i_2,\ldots,i_k)=\mathbf{u}(i_1,i_2,\ldots,i_{k-2},\mathbf{r}(i_{k-1},i_k)) \qquad \text{and}$$

$$\mathbf{m}(i_1,i_2,\ldots,i_k)=[\mathbf{u}(i_1,i_2,\ldots,i_{k-2},\mathbf{r}(i_{k-1},i_k))-1]n^{k-1}+\mathbf{m}(i_1,i_2,\ldots,i_{k-2},\mathbf{s}(i_{k-1},i_k)).$$

In four steps we prove:

(a) $\mathbf{U}_n^k$ is a Latin $\mathbf{H}_n^k$,
(b) hyperedges of $\mathbf{M}_n^k$ are labels from the set $\{1,2,\ldots,n^k\}$,
(c) no two labels of hyperedges of $\mathbf{M}_n^k$ are equal,
(d) sums of labels of hyperedges incident with $(k-1)$ vertices $\mathbf{M}_n^k$ are equal.

(a) Because $\mathbf{R}_n$ is a Latin square both sets

$$\{\mathbf{r}(x,i_k):x=1,2,\ldots,n\} \qquad \text{and} \qquad \{\mathbf{r}(i_{k-1},x):x=1,2,\ldots,n\}$$

are equal to the set $\{1,2,\ldots,n\}$ and therefore

$$\{\mathbf{u}(i_1,i_2,\ldots,i_{k-2},\mathbf{r}(x,i_k):x=1,2,\ldots,n\} \qquad \text{and}$$

$$\{\mathbf{u}(i_1,i_2,\ldots,i_{k-2},\mathbf{r}(i_{k-1},x):x=1,2,\ldots,n\}$$

are equal to the set $\{1,2,\ldots,n\}$. Because $\mathbf{U}_m^{k-1}$ is a Latin hypergraph it follows that the labels of

$$\{\mathbf{u}(i_1,i_2,\ldots,i_{j-1},x,i_{j+1},\ldots,i_{k-2},\mathbf{r}(i_{k-1},i_k)):x=1,2,\ldots,n\}$$

is equal to the set $\{1,2,\ldots,n\}$ for $j=1,2,3,\ldots,k-2$.

(b) All labels of hyperedges of $\mathbf{U}_n^k$ are from the set $\{1,2,\ldots,n\}$ and all labels of hyperedges of $\mathbf{M}_n^{k-1}$ are from the set $\{1,2,\ldots,n^{k-1}\}$. It follows immediately that for all labels of hyperedges of $\mathbf{M}_n^k$ we have

$$1\le\mathbf{m}(i_1,i_2,\ldots,i_k)\le n^k \qquad \text{for all} \qquad 1\le i_1,i_2,\ldots,i_k\le n.$$

(c) Let us suppose that $\mathbf{m}(i_1,i_2,\ldots,i_k)=\mathbf{m}(i_1',i_2',\ldots,i_k')$. We show that this implies $(i_1,i_2,\ldots,i_k)=(i_1',i_2',\ldots,i_k')$.

From the definition of $\mathbf{M}_n^k$ it follows

$$\begin{aligned}&[\mathbf{u}(i_1,i_2,\ldots,i_{k-2},\mathbf{r}(i_{k-1},i_k))-1]n^{k-1}+\mathbf{m}(i_1,i_2,\ldots,i_{k-2},\mathbf{s}(i_{k-1},i_k))=\\&[\mathbf{u}(i_1',i_2',\ldots,i_{k-2}',\mathbf{r}(i_{k-1}',i_k'))-1]n^{k-1}+\mathbf{m}(i_1',i_2',\ldots,i_{k-2}',\mathbf{s}(i_{k-1}',i_k')).\end{aligned}$$

By rearranging this equality we get

$$\begin{aligned}&[\mathbf{u}(i_1,i_2,\ldots,i_{k-2},\mathbf{r}(i_{k-1},i_k))-\mathbf{u}(i_1',i_2',\ldots,i_{k-2}',\mathbf{r}(i_{k-1}',i_k'))]n^{k-1}=\\&\mathbf{m}(i_1',i_2',\ldots,i_{k-2}',\mathbf{s}(i_{k-1}',i_k'))-\mathbf{m}(i_1,i_2,\ldots,i_{k-2},\mathbf{s}(i_{k-1},i_k)).\end{aligned} \tag{1}$$

The left hand side of (1) is a multiple of n^{k-1} and the right side is a difference of labels of two hyperedges of $\mathbf{M}_n^{k-1}$ which is not a non-zero multiple of n^{k-1}. From (1) it follows that

$$\mathbf{u}(i_1, i_2, \dots, i_{k-2}, \mathbf{r}(i_{k-1}, i_k)) = \mathbf{u}(i'_1, i'_2, \dots, i'_{k-2}, \mathbf{r}(i'_{k-1}, i'_k)) \tag{2}$$

and

$$\mathbf{m}(i_1, i_2, \dots, i_{k-2}, \mathbf{s}(i_{k-1}, i_k)) = \mathbf{m}(i'_1, i'_2, \dots, i'_{k-2}, \mathbf{s}(i'_{k-1}, i'_k)). \tag{3}$$

In $\mathbf{M}_n^{k-1}$ no two hyperedges have the same labels and therefore from (3) it follows that

$$i_1 = i'_1, \quad i_2 = i'_2, \dots, \; i_{k-2} = i'_{k-2} \qquad \text{and}$$

$$\mathbf{s}(i_{k-1}, i_k) = \mathbf{s}(i'_{k-1}, i'_k). \tag{4}$$

By substitution of the first $(k-2)$ equalities in (2) we get

$$\mathbf{u}(i_1, i_2, \dots, i_{k-2}, \mathbf{r}(i_{k-1}, i_k)) = \mathbf{u}(i_1, i_2, \dots, i_{k-2}, \mathbf{r}(i'_{k-1}, i'_k)). \tag{5}$$

Because $\mathbf{U}_n^{k-1}$ is a Latin hypergraph, from the equality of the first $(k-2)$ indices in (5) it follows that

$$\mathbf{r}(i_{k-1}, i_k) = \mathbf{r}(i'_{k-1}, i'_k). \tag{6}$$

From the assumption that $\mathbf{R}_n$ and $\mathbf{S}_n$ are orthogonal Latin squares and (4), (6) we get $i_{k-1} = i'_{k-1}$ and $i_k = i'_k$ which completes (c).

(d) For every $j = 1, 2, \dots, k$ we have

$$\sum_{i_j=1}^{n} \mathbf{m}(i_1, i_2, \dots, i_{k-2}, i_{k-1}, i_k) = \sum_{i_j=1}^{n} [\mathbf{u}(i_1, i_2, \dots, i_{k-2}, \mathbf{r}(i_{k-1}, i_k)) - 1] n^{k-1} +$$

$$\sum_{i_j=1}^{n} \mathbf{m}(i_1, i_2, \dots, i_{k-2}, \mathbf{s}(i_{k-1}, i_k)) = [\tfrac{n(n+1)}{2} - n] n^{k-1} + \tfrac{n(n^{k-1}+1)}{2} = \tfrac{n(n^k+1)}{2}.$$

This completes the proof.

Remark. The above construction of $\mathbf{M}_n^k$ is based on the use of two orthogonal Latin squares and therefore it is not valid for $n = 2, 6$.

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